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TEACHING MATERIAL ON

MATHEMATICS

SCHOOL OF SCIENCE

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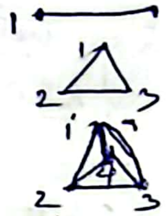
Linear Programming
The Simplex Method

Most of the real life problems when formulated to LP-problem have more than two variables and therefore by graphical ~~solution~~ method we can't find the solution then rather an efficient method is required to provide an optimal solution called as 'Simplex Method' for solving an LP problem. The method was first developed by G. B. Dantzig in 1947.

Simplex is a mathematical term represents an object in n-dimensional space connecting n+1 points.

In one dimension a simplex is a line segment connecting two points, in two dimension, it is a triangle formed by joining three points, in three dimensions, it is a four sided pyramid having four corners.



and so on

In graphical method
the extreme points of the feasible solution space region are examined to search for optimal soln at one of them.

Whereas, for LP problem with several variables we may not be able to graph the feasible region, but the optimal soln will still lie at an extreme point of the many-sided, multidimensional figure (called an n-dimensional polyhedron) that represents the feasible solution space.

Since the extreme points and feasible solution space are required to be searched on n-dimensional polyhedron so an efficient algorithm is required to follow until the required optimal soln is not found.

... (1/0). we have got — ve values at three places

$\Delta_3 = c_3 - c_B Y_3 = 4-(0,0,0)(0,5,4) = 4$

Since the no. of extreme points (corners or vertices) of feasible solution space are clearly finite, the method assures an improvement in the value of objective function as we move from one iteration (extreme point) to another and achieve optimal solution in a finite number of steps and also indicates when an unbounded solution is reached.

§1. Standard form of an L.P. problems:—

Simplex method consists in converting a LP problem into its standard form with following characteristics:→

(i) All the constraints should be expressed as equations by adding slack or surplus and/or artificial variables.

(ii) The right-hand side of each constraints should be made non-negative; if it is not non-negative, this should be done by multiplying both sides of the resulting constraints by −1.

(iii) The objective function should be of maximization type.

The standard form of an LP problem is expressed as

Optimize (Maximize or Minimize)

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n + 0s_1 + 0s_2 + \cdots + 0s_m$$

Subject to the linear constraints

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n + 1s_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n + 0s_1 + 1s_2 &= b_2 \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n + \cdots + 1s_m &= b_m \end{aligned}\right\} \quad (1)$$

and $x_1, x_2, x_3, \ldots, x_n, s_1, s_2, s_3, \ldots, s_m \geq 0$

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In matrix form the standard form can be expressed as:

Optimize (Maximize or Minimize)

$$Z = \bar{c}\bar{x} + 0\bar{s}$$

Subject to the constraints

$$A\cdot\bar{x} + \bar{s} = \bar{b}$$

and $\bar{x}, \bar{s} \geq 0$

where, $\bar{c} = (c_1, c_2, \cdots c_n)$ is the row vector

$\bar{x} = (x_1, x_2, \cdots x_n)^T$ is the column vector

$\bar{b} = (b_1, b_2 \cdots b_m)^T$ is the column vector

$\bar{s} = (s_1, s_2 \cdots s_m)^T$ are column vector

and
$$\bar{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ - & - & - & & - \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

is $m\times n$ matrix of coefficients of variables $x_1, x_2 \cdots x_n$ in the constraints.

N.B. There are three types of additional variables, namely

(i) Slack variables (s)

(ii) Surplus variables ($-s$)

and (iii) Artificial variables (A)

are ~~generally~~ added in the given LP problem to convert it into the standard form for the following reason:—

(a) These variables allows us to convert inequalities into equalities, thereby converting the given LP problem into a form that is amenable to algebraic treatment (or solution).

the (≥ 0). we have got $-ve$ values at three places

$\Delta_3 = c_3 - c_B Y_3 = 4 - (0,0,0)(0,5,4) - 1$

(b) These variables permit us to make a more comprehensive & economic interpretation of a final solution.

(c) Help us to get an initial feasible solution represented by the columns of the identity matrix.

A summary of the extra variables to be added in the given LP problem to convert it into a standard form is given in the following table:—

Table - 1

Type of Constraints	Extra Variable needed	Coefficients of Extra Variables to the objective function (Min Z)	(Max Z)	Presence of Extra variables in the Initial soln
i) Less than or equal to (≤)	A slack variable is added	0	0	yes
ii) Greater than or equal to (≥)	A surplus variable is subtracted	0	0	No.
	An Artificial variable is added	+M	−M	yes
iii) Equal to (=)	Only artificial variable is added	+M	−M	yes

Remark:- I) A slack variables represents unused resource, either in the form of time on a machine, labour hours, money, warehouse space or any types / kind / number of such resources in various business problems. Since these variables yield no profit, therefore such variables are added to the original objective function with zero coefficients.

II) A surplus variables represents amount by which solution values exceeds a resource also called as negative slack variables and like slack variables these variables also carry a zero coefficients in the objective function.

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Simplex Algorithm (Maximization Case)

③

~~For a LP problem~~ In order to find an optimal solution (if it exists) to a LP problem following are the steps of the Simplex Algorithm :⟶

Step-1 Formulation of the Mathematical Model :-

1) Formulate the mathematical model of the given linear programming problem.

2) If the objective function is of minimization type, then convert it into one of maximization type by using the following relationship.

$$\text{Minimize } Z = -\text{ Maximize } Z^*$$

where $Z^* = -Z$.

(3) Check whether all the b_i $(i=1,2,\dots m)$ values are positive, else multiply by -1 and make $b_i \geq 0$ (remember to change the $\leq$ to $\geq$ sign and vice-versa).

(4) Express the mathematical model of the LP problem in the standard form by adding additional variables in the left side of each constraints and assign a zero-cost coefficient to these variables in the objective function.

(5) Replace each unrestricted variables with the difference of two non-negative variables, replace each non-positive variable with a new non-negative variable whose value is the negative of the original variable.

Step-2. Set-up the Initial Solution:—

Write down the coefficients of all the variables in the LP model in the tabular form, as shown in the table below to get an initial basic feasible solution $[X_B = B^{-1}b]$

....., ——— | tve (≥ 0). we have got -ve values at three places

$\Delta_3 = c_3 - c_B Y_3 = 4 - (0,0,0)(0,5,4)$

n-variables
& m(<n) constraints

(TD) Table: Initial Simplex Table

bfs – algebraic of extreme

1st row is (cost or profit/unit to Z of each of the variables) $C_j \rightarrow$ used to determine the variable to be entered basis matrix B:

			c_1	c_2	----	c_n	0	0	...
Coefficient of Basic Variables (C_B)	Variables in Basic B	Value of Basic variables $\bar{b}\,(=\bar{x}_B)$	← $(n-m)$ Non-basic x_1	Variables x_2	--- → x_n	(m) ← basic s_1	s_2	...	s_m
C_{B1}	s_1	$x_{B_1} = b_1$	a_{11}	a_{12}	--- a_{1n}	1	0	0	... 0
C_{B2}	s_2	$x_{B_2} = b_2$	a_{21}	a_{22}	--- a_{2n}	0	1	0	
⋮	⋮	⋮	—	—	—	—	—		
C_{Bm}	s_m	$x_{B_m} = b_m$	a_{m1}	a_{m2}	a_{mn}	0	0	0	
$Z = \sum C_{B_i} x_{B_i}$ = (B.V. Coeff × Value of B.V.)	$Z_j = \sum C_{B_i} x_{ij}$ = $\sum$ (B.V. Coeff. × (jth column of data matrix))		0	0	0	0	0	0	0
	$C_j - Z_j$		$c_1 - z_1$	$c_2 - z_2$	-- $c_n - z_n$	0	0	0	.. 0

we assign values of b_i's to the column variables in the identity matrix because $[\bar{x}_B = B^{-1}b = Ib = b]$

basis matrix = Identity matrix is basis matrix

$\therefore B = I$

$\therefore B^{-1} = I$

is a square matrix. Its size depends upon no. of constraints.

A bfs obtained by picking one basic variable for each constraint and remaining ones are non-basic and have zero values. However in certain cases some basic variables may have zero values. This situation is called degeneracy and is discussed later.

The second row is the major column headings for the Simplex table. Column 'C_B' list the coeff. of the current basic variables in the objective fn. These values are used to calculate value of Z when one unit of any variable is brought in the solution. Column headed by x_B represents the current values of the corresponding variables in the basis.

The identity matrix (or basis matrix) represents the slack variable which have been added to the constraints and each column of the identity matrix also represents basic variables to be listed in column B.

Number a_{ij} in columns under each variable are also called "substitution rate" or exchange coefficients because these represents the rate at which resources i ($i=1,2,\dots m$) is consumed by each unit of an activity j ($j=1,2,\dots n$).

The values z_j represents the amount by which the value of objective function Z would be decreased or increased, if one unit of given variable is added to the new solution. Each of the value in the $c_j - z_j$ row represents the net amount of increase (or decrease) in the objective function that would occur when one unit of variable represented by the column head is introduced into the solution. That is $c_j - z_j$ (net effect)

= c_j (incoming unit profit/cost) − z_j (outgoing total profit/cost)

where z_j = coefficient of basic variables column × Exchange coefficient column

Step-3. Test for optimality:—

Calculate the $c_j - z_j$ value for all non-basic variables. To obtain the value of z_j multiply each element under variables-column head (column, a_j of the coefficient matrix) with the corresponding elements under coefficients of basic variables column (c_B column). Examine the value of $c_j - z_j$. There may be three cases.

(i) If all $c_j - z_j \le 0$, then the basic feasible solution is optimal.

(ii) If at least one column of the coefficients matrix i.e. (a_{ij}) for which $c_j - z_j > 0$ and all elements are negative (i.e. $a_{ik} < 0$), then there exists an unbounded solution to the given problem.

(iii) If at ~~least~~ least one $c_j - z_j > 0$ and each of these has at least one positive element (i.e. a_{ij}) for some row, then it indicates that an improvement in the value of objective function Z is possible.

The (>0), we have got +ve values at three places

$\Delta 3 = c_3 - c_B Y_3 = 4 - (0,0,0)(0,5,4) = 4$

Step-4. Select the variable to Enter the Basis:-

If Case (iii) of Step-3 holds, then select a variable that has the largest $z_j - c_j$ value to enter into the new solution. That is

$$c_k - z_k = \text{Max}\left((c_j - z_j) : c_j - z_j > 0 \right).$$

The column to be entered is called the key or pivot column. Obviously, such a variable ~~method~~ indicates the largest per unit improvement in the current solution. ~~Obviously, such a variable indicates.~~

Step 5:- Test for feasibility (variable to leave the basis):— After identifying the variable to become basic variable, the variable to be removed from the existing set of basic variables is determined. For this, each number in x_B-column (i.e., b_i values) is divided by the corresponding (but positive) number in the key column and a row is selected for which this ratio, [(constant column)/(key column)] is non-negative and minimum. This ratio is called the replacement (exchange) ratio.

That is $$\frac{x_{Br}}{a_{rj}} = \text{Min}\left\{ \frac{x_{Bi}}{a_{rj}} \; ; a_{rj} > 0 \right\}.$$

This ratio limits the number of units of incoming variable that can be obtained from the exchange. It may be noted here that division by negative or zero element in key column is not permitted.

The row selected in this manner is called the key or pivot row and represents the variable which will leave the solution.

The element that lies in the intersection of the key row and key column of the simplex table is called key or pivot element.

Step-6 Finding the New solution:—

(i) If the Key element is 1, then the row remains the same in the new simplex table.

(ii) If the Key element is other than 1, then divide each element in the Key row (including element in x_B-column) by the Key element, to find the new values for that row.

(iii) The new values of the elements in the remaining rows for the new simplex table can be obtained by performing elementary row operations on all rows so that all elements except the Key element in the key column are zero.

In other words, for each row other than the key row, we use the formula:—

$$\text{Number in new row} = \begin{pmatrix}\text{No in}\\ \text{old row}\end{pmatrix} \pm \left[\begin{pmatrix}\text{No. above}\\ \text{or below}\\ \text{Key element}\end{pmatrix} \times \begin{pmatrix}\text{corresponding}\\ \text{no. in the new}\\ \text{row, that is}\\ \text{replaced}\\ \text{in step 6 (ii)}\end{pmatrix}\right]$$

The new entries in C_B (coefficient of basic variables and x_B (value of basic variables) column is updated in the new simplex table of the current solution.

Step-7 Repeat the procedure

Go to step 3 and repeat the procedure until all entries in the $C_j - Z_j$ row are either negative or zero.

Example-4.1:— Use the simplex method to solve the following LP problem:—

Maximize $Z = 3x_1 + 5x_2 + 4x_3$

Subject to the constraints,

$$2x_1 + 3x_2 \le 8$$
$$2x_2 + 5x_3 \le 10$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$

and $x_1, x_2, x_3 \ge 0$

..., ——— | The (≥0). we have got —ve values at three plac

Now, $\Delta_3 = c_3 - C_B Y_3 = 4 - (0,0,0)(0,5,4) = 4$

Sol. Step-1 Introducing non-negative slack variables s_1, s_2 and s_3 to convert in equality constraints to equality. Then the LP problem becomes.

Maximize $Z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to the constraints

$$2x_1 + 3x_2 + 0x_3 + 1s_1 + 0s_2 + 0s_3 = 8$$
$$0x_1 + 2x_2 + 5x_3 + 0s_1 + 1s_2 + 0s_3 = 10$$
$$3x_1 + 2x_2 + 4x_3 + 0s_1 + 0s_2 + 1s_3 = 15$$

and $x_1, x_2, x_3 \geq 0$.

Step-2. (i) Since all b_i ($i=1,2,3$) (RHS values) are +ve (non-negative ≥ 0)

(ii) We can choose initial basic feasible solution as

$x_1 = x_2 = x_3 = 0$; $s_1 = 8$, $s_2 = 10$, $s_3 = 15$. and Max $Z = 0$

This solution can also be read from the initial simplex Table-1 by equating row wise values in the basis (B) column and solution values (x_B) column.

Table 1 Initial simplex table

Profit per unit $\bar{C}_B$	Variables in Basic B	Solution values $b = \bar{x}_B$	$c_j \rightarrow$ 3	5	4	0	0	0	Min exact ratio
			x_1	x_2	x_3	s_1	s_2	s_3	
			Non-basic			basic			
0	s_1	8	2	3 (key elt.)	0	1	0	0	8/3 (Min)
0	s_2	10	0	2	5	0	1	0	10/2
0	s_3	15	3	2	4	0	0	1	15/3
$Z = 0$		Z_j	0	0	0	0	0	0	
		$z_1 = 0(2)+0(0)+0(3) = 0$	z_j = (Basic Var Coeff, $\bar{C}_B$) × (jth column of data matrix)						
$Z = \Sigma \bar{C}_B \bar{x}_B$		$c_j - Z_j$	3	5 ↑ (Max ≥ 0)	4	0	0	0	

Step-3 To see whether the current solution given in table-1 is optimal or not, calculate

net profit $= c_j - z_j = c_j - \bar{C}_B B^{-1} \bar{a}_j = c_j - C_B y_j$, $j = 1,2,3$ for non-basic variables

from introducing one unit of each variable x_1, x_2, x_3 into the new solution mix

$c_1 - z_1 = 3 - 0 = 3$
$c_2 - z_2 = 5 - 0 = 5$
$c_3 - z_3 = 4 - 0 = 4$

∴ z_j and $c_j - z_j$ rows are added into the table-1

Also, Z = (Basic Var Coeff; C_B) × (Basic Var value $\bar{x}_B$)
$= 0(8) + 0(10) + 0(15) = 0$.

Since all $c_j - z_j \geq 0$ ($j = 1,2,3$) (8), the current solution is (6) not optimal. Variable x_2 is choosen to enter into the basis as $c_2 - z_2 = 5$ is the largest +ve (≥ 0) no. in the x_2-Column, where all element are +ve (≥ 0)
$\Rightarrow$ every unit of variable x_2, the objective fn will increase in value by 5. Hence, x_2-Column is the key column.

Step-4 The variable to leave the basis is determined by dividing the values in the X_B-column by the corresponding elements in the key column as shown in the ~~next~~ table-1. Since the Min exchange ratio 8/3 out of 8/3, 10/2, 15/2 is minimum & lies in row-1, therefore the basic variable S_1 is choosen to leave the solution (basis)

Step-5. (Iteration-1) Since the key element enclosed in Table-1 is not 1, we divide all elements of the key row by 3 to obtain new values of the elements in this row.
Also the new values of the elements in the remaining rows for the new table-2 are obtained by performing the following elementary row ~~transformations~~ operations on all other rows so that all the elements except the key element 1 in the key column are zero

For key row → $R_1(\text{new}) \to R_1(\text{old}) \div 3$ (key element)

$$\to \left(8/3,\ 2/3,\ 3/3,\ 0/3,\ 1/3,\ 0/3,\ 0/3\right)$$
$$= \left(8/3,\ 2/3\ 1,\ 0\ ,1/3,\ 0,\ 0\right)$$

For all other rows → $R_2(\text{new}) \to R_2(\text{old}) - 2R_1(\text{new})$; $R_3(\text{new}) \to R_3(\text{old}) - 2R_3(\text{new})$

$\frac{14}{3}$	$\to 10 - 2\times\frac{8}{3} = \frac{30-16}{3}$	$\frac{29}{3}$	$\to 15 - 2\times\frac{8}{3} = \frac{45-16}{3} = \frac{29}{3}$
$-4/3$	$\to 0 - 2\times\frac{2}{3} = 0 - 4/3$	$5/3$	$\to 3 - 2\times\frac{2}{3} = \frac{9-4}{3} = \frac{5}{3}$
0	$\to 2 - 2\times1 = 0$	0	$\to 2 - 2\times1 = 0$
5	$\to 5 - 2\times0 = 5$	4	$\to 4 - 2\times0 = 4$
$-2/3$	$\to 0 - 2\times1/3 = -2/3$	$-2/3$	$\to 0 - 2\times1/3 = -2/3$
1	$\to 1 - 2\times0 = 1$	0	$\to 0 - 2\times0 = 0$
0	$\to 0 - 2\times0 = 0$	1	$\to 1 - 2\times0 = 1$

I < (≥ 0). we have got -ve values at three places

Δ_{3}. $\Delta 3 = c_3 - c_B Y_3 = 4 - (0,0,0)(0,5,4) = 4$

Table-2 Improved solution

C_B Profit per unit	x_B variables in basis	b ($=x_B$) solution values	($C_j \to$) 3 x_1	5 x_2	4 x_3	0 s_1	0 s_2	0 s_3	Min Ratio x_B/x_3
5	x_2	8/3	2/3	1	0	1/3	0	0	$\times$
0	s_2	14/3	$-4/3$	0	5	$-2/3$	1	0	$(14/3)/5 = 14/15$
0	s_3	29/3	2/3	0	4	$-2/3$	0	1	$(29/3)/4 = 29/12$
$Z = \sum C_B x_B = 40/3$		$Z_j = \sum C_B \cdot y_j$	$Z_1 = 10/3$	5	0	5/3	0	0	
		$C_j - Z_j$	$\left(3 - \frac{10}{3} = -\frac{1}{3}\right)$	0	4 ↑ Max(+ve)	$-5/3$	0	0	

An improved basic feasible solution can be read from table-2 as $x_2 = 8/3$, $s_2 = 14/3$, $s_3 = 29/3$ and $x_1 = s_1 = x_3 = 0$ as Max $Z = 40/3$
$(= 5(8/3) + 0(14/3) + 0(29/3))$

Let us check whether the solution obtained in table 2 is an optimal or not by checking whether $C_j - Z_j$ is either negative or zero. But here in table 1 $C_3 - Z_3 = 4$ (>0) & Maximum. Therefore, we have to proceed further as earlier.

Step-6 (Iteration 2): We repeat step 3 to 5. Table-3 is obtained by performing following row operation to enter variable x_3 into the basis and to drive out s_2 from the basis.

$R_2(\text{new}) = R_2(\text{old}) \div (\text{key element})\ 5$

$R_2(\text{new}) = \frac{14}{3} \div 5 = \frac{14}{15}$ $\quad \therefore \left(\frac{14}{15}, -\frac{4}{15}, 0, 1, -\frac{2}{15}, \frac{1}{5}, 0\right)$

$-\frac{4}{3} \div 5 = -\frac{4}{15}$

$0 \div 5 = 0$

$\frac{5}{5} = 1$

$-\frac{2}{3} \div 5 = -2/15$

$1 \div 5 = \frac{1}{5}$

$0 \div 5 = 0$

New $R_1 = R_1 - 0 R_2 = R_1$

$R_3 = R_3 - 4R_2$

$= \frac{29}{3} - 4\frac{14}{15} = \frac{89}{15}$

$\Rightarrow \frac{2}{3} - 4\left(-\frac{4}{15}\right) = \frac{26}{15}$

$\Rightarrow 0 - 4(0) = 0$

$\Rightarrow 4 - 4 \times 1 = 0$

$-2/3 - 4\left(-\frac{2}{15}\right) = -\frac{2}{15}$

$0 - 4\left(\frac{1}{5}\right) = -\frac{4}{5}$

$0 - 4(0) = 0$

$(\text{New } R_3) = (\text{Old } R_3) - 5(R_2 \text{ new})$

$\rightarrow \frac{22}{3} - 5\left(\frac{14}{15}\right) =$

$\rightarrow \frac{5}{3} - 5\left(-\frac{4}{15}\right) =$

$\rightarrow 0 - 5(0)$

$\rightarrow 4 - 5(1)$

$\rightarrow \frac{2}{3} - 5\left(-\frac{2}{15}\right)$

$\rightarrow 0 - 5\left(\frac{1}{5}\right)$

$\rightarrow 1 - 5(0)$

$Z_1 = 5\left(\frac{2}{3}\right) + 4\left(-\frac{4}{15}\right) + 0\left(\frac{41}{15}\right) = \frac{34}{15}$

$Z_2 = 5(1) + 4(0) + 0(0) = 5$

$Z_3 = 5(0) + 4(1) + 0(0) = 4$

$Z_4 = 5(1/3) + 4\left(-\frac{2}{15}\right) + 0\left(-\frac{2}{15}\right) = \frac{17}{15}$

$Z_5 = 5(0) + 4(1/5) + 0(-4/5) = \frac{4}{5}$

Table-3. Improved Solution

Coeff. basic variables C_B	Variables in basis B	Solution values ($X_B = b$)	x_1	x_2	x_3	s_1	s_2	s_3	Min Exchange Rate
		$C_j \rightarrow$	3	5	4	0	0	0	
5	x_2	8/3	2/3	1	0	1/3	0	0	×
4	x_3	14/15	$-\frac{4}{15}$	0	1	$-\frac{2}{15}$	$\frac{1}{5}$	0	×
0	s_3	89/15	$\frac{41}{15}$	0	0	$-\frac{2}{15}$	$-\frac{4}{5}$	1	$\frac{89}{15} \div \frac{41}{15} \rightarrow$
$Z = \sum C_B \cdot X_B = 256/15$		Z_j	$\frac{34}{15}$	5	4	17/15	4/5	0	
		$Z_1 = 5(2/3) + 4(-4/15) + 0(41/15) = 34/15$							
		$C_j - Z_j$	11/15 ↑ Max(+ve)	0	0	−17/15	−4/5	0	

Iteration 3. A new table Table-4 is constructed for improved solution because $\frac{11}{15}$ is the largest non-negative value of $C_j - Z_j$, so performing row operations in the next table we get

$R_3(\text{new}) \rightarrow R_3(\text{Old}) \div (\text{key element})$

$\rightarrow \left[\left(89/15 \times \frac{15}{41}\right), \left(\frac{41}{15} \times \frac{15}{41}\right), 0, 0, \left(\frac{-2}{15} \times \frac{15}{41}\right), \left(-\frac{4}{5} \times \frac{15}{41}\right), \frac{15}{41}\right]$

$= \left[\frac{89}{41}, 1, 0, 0, -\frac{2}{41}, \frac{-12}{41}, \frac{15}{41}\right]$

Tre (≠0). we have got −ve values at three places

$\Delta_3 = c_3 - c_B Y_3 = 4 - (0, 0, 0)(0, 5, 4) = 4$

$R_1(\text{new}) \to R_1(\text{old}) - (2/3)(R_3\ \text{new})$

$8/3 - (2/3) \times (89/3) = 50/41$

$2/3 - 2/3 \times 1 = 0$

$1 - 2/3 \times 0 = 1$

$0 - 2/3 \times 0 = 0$

$1/3 - 2/3 \times (-2/41) = 15/41$

$0 - 2/3 \times (-12/41) = 8/41$

$0 - 2/3 \times (15/41) = -10/41$

$R_2(\text{new}) \to R_2(\text{old}) + \left(\frac{4}{15}\right) R_3(\text{new})$

$14/15 + 4/15 \times \frac{89}{41} = 62/41$

$-4/15 + \frac{4}{15} \times 1 = 0$

$0 + 4/15 \times 0 = 0$

$1 + 4/15 \times 0 = 1$

$-2/15 + 4/15 \times (-\frac{2}{41}) = -6/41$

$1/5 + 4/15 \times (-\frac{12}{41}) = 5/41$

$0 + 4/15 \times 15/41 = 4/41$

Table ~~IV~~ optimal solution.

Coefficient Basic variables or Profit/unit C_B	Variable in Basis B	$C_J \to$ Solution values $b (= X_B)$	3 x_1	5 x_2	4 x_3	0 s_1	0 s_2	0 s_3	Min Exp rate
5	x_2	50/41	0	1	0	15/41	8/41	-10/41	✗
4	x_3	62/41	0	0	1	-6/41	5/41	4/41	✗
3	x_1	89/41	1	0	0	-2/41	-12/41	15/41	✗
$Z = \sum C_B \cdot X_B$		Z_J	1	5	4	45/41	24/41	11/41	
$= 765/41$		$C_J - Z_J$	0	0	0	-45/41	-24/41	-11/41	

In this table all $C_J - Z_J \leq 0$ for non-basic variables. Therefore the optimal solution is reached with $x_1 = \frac{89}{41}$, $x_2 = \frac{50}{41}$ and $x_3 = \frac{62}{41}$ and the optimum value of $Z = 765/41$ Ans

Linear P. P. (Using Big M. Method) or Modified LPP

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Simplex Method $\leq$: add slack variables $(+S)$

Big M Method $\geq$: Sub surplus variable $(-S)$ and add Artificial variables $(+A)$

Problem. [we will use same simplex algo in Big M method]

Sl. Solve the following LPP using Big M. Method :-

Minimize $Z = 7x_1 + 15x_2 + 20x_3$

s.t. $2x_1 + 4x_2 + 6x_3 \geq 24$

$3x_1 + 9x_2 + 6x_3 \geq 30$

$x_1, x_2, x_3 \geq 0$.

Soln. Subtract surplus variables and add Artificial variables (Max^m +ve value)

Min $Z = 7x_1 + 15x_2 + 20x_3 + \overset{+ve\ zero}{0S_1 + 0S_2} + MA_1 + MA_2$

s.t. $2x_1 + 4x_2 + 6x_3 - S_1 + A_1 = 24$

$3x_1 + 9x_2 + 6x_3 - S_2 + A_2 = 30$

$x_1, x_2, x_3, S_1, S_2, A_1$ and $A_2 \geq 0$

Note- Artificial variables are included to solve the model and in final solution these Artificial variables should not be available.

Initial Table :- key elt.

	C_j	7	15	20	0	0	M	M		
CB.	B.V.	x_1	x_2	x_3	S_1	S_2	A_1	A_2	Soln	Ratio
R_1 M	A_1	2	4	6	-1	0	1	0	24	$\frac{24}{4}$
R_2 M	A_2	3	9	6	0	-1	0	1	30	$\frac{30}{9} = \frac{10}{3}$ →
	Z_j	5M	13M	12M	-M	-M	M	M	54M	
	✓ $C_j - Z_j$	7-5M	15-13M ↑	20-12M	M	M	0	0		

$$Z_j = \sum_{i=1}^{2} (C_{B_i})(a_{ij})$$

or

(1) CB of R_1 × 1st Element + CB of R_2 × 1st element.

$M \times 2 + 3 \times M = 5M$

$M \times 4 + M \times 9 = 13M$ etc.

(2) $C_j - Z_j$

(3) check Optimality condition for Minimization

$C_j - Z_j \geq 0$: all the $C_j - Z_j$ values may either be 0 or +ve value ≥ 0

Here we have negative values 7-5M, 15-13M, 20-12M - (a) amongst these select most negative value

clearly 15-13M is the most negative value

(a) Find key column ∴ x_2 column is our key column.

(b) Fill Ratio Column to least +ve value of ratio to find 'key row'.

clearly $24/4 = 6$ & $30/9 = \frac{10}{3}$ least

(c) We need to prepare the 1st Iteration. Here V_B & C_B remains same in the previous table.

Iteration – I (least +ve value)

		C_j	7	15	20	0	0	M	M		
	CB	BV	x_1	x_2	x_3	S_1	S_2	A_1	A_2	soln	Ratio
R_3	M	A_1	$\frac{2}{3}$	0	$\frac{10}{3}$	-1	$\frac{4}{9}$	1	—	$32/3$	$\frac{32/3}{10/3} = \frac{16}{5}$ →
R_4	15	x_2	$1/3$	1	$\frac{2}{3}$	0	$-1/9$	0	—	$10/3$	$\frac{10/3}{2/3} = 5$
		Z_j	$\frac{2}{3}M+5$	15	$\frac{10}{3}M+10$	-M	$\frac{4}{9}M - \frac{5}{3}$	M	—	$\frac{32}{3}M + 50$	
		$C_j - Z_j$	$-\frac{2}{3}M+2$	0	$-\frac{10}{3}M+10$ ↑	M	$-\frac{4}{9}M + \frac{5}{3}$	0	—		

Most negative value

(d) for R_3 there is a formula :-

$$R_3 = \text{old value} - \frac{\text{Key Column value} \times \text{key row value}}{\text{key elt (value)}}$$

or

old value – (key column value × R_4) ∵ $R_4 = \frac{\text{key row val}}{\text{key value}}$

$2 - \frac{4 \times 3}{9} =$

i) ✓ $2 - 4 \times \frac{1}{3} = 2 - \frac{4}{3} = \frac{6-4}{3} = \frac{2}{3}$

||ly others, ii) $4 - 4 \times \frac{9}{9} = 0$

iii) $6 - 4 \times \frac{2}{3} = 6 - \frac{8}{3} = \frac{18-8}{3} = \frac{10}{3}$

iv) $-1 - 4 \times 0 = -1$

v) $0 - 4 \times (-\frac{1}{9}) = \frac{4}{9}$

vi) $1 - 4 \times 0 = 1$

Soln → $24 - 4 \times \frac{10}{3} = 24 - \frac{40}{3} = \frac{72-40}{3} = \frac{32}{3}$

(e) Next to find Z_j $M \times \frac{2}{3} + 15 \times \frac{1}{3} = \frac{2}{3}M + 5$

(f) Next find $C_j - Z_j = 7 - \frac{2}{3}M - 5 = 2 - \frac{2}{3}M$

(g) Check the optimality condition. Here for minimization all $C_j - Z_j$ values must be either 0 or +ve (≥ 0). We have got –ve values at three places

$\Delta_3 = C_3 - C_B Y_3 = 4 - (0, 0, 0)(0, 5, 4) = 4$

Key coln is x_3

Key row is find by getting least +ve ratio so $\frac{16}{5}$ is least +ve value
∴ A_1 is the key row & x_3 is the key column ∴ x_3 is entering and A_1 is leaving variable, C_B, BV will be same.

So we proceed for next Iteration 2

	C_j	7	15	20	0	0	M	M		
C_B	BV	x_1	x_2	x_3	S_1	S_2	A_1	A_2	Soln	Ratio
R_5 20	x_3	$\frac{2/3}{10/3}=\frac{1}{5}$	$0\div\frac{10}{3}=0$	1	$-1\div\frac{10}{3}=\frac{-3}{10}$	$\frac{2}{15}$	–	–	$\frac{16}{5}$	
R_6 15	x_2	$\frac{1}{5}$	1	0	$\frac{1}{5}$	$-\frac{1}{5}$	–	–	$\frac{6}{5}$	
	Z_j	7	15	20	-3	$-\frac{1}{3}$	–	–	82	
	C_j-Z_j	0	0	0	3	1/3				

∴ For Row 6 = Old value – (Key column value × R_5)

$$\frac{1}{3}-\left(\frac{2}{3}\times\frac{1}{5}\right)=\frac{1}{3}-\frac{2}{15}=\frac{5-2}{15}=\frac{3}{15}=\frac{1}{5}$$
$$1-\left(\frac{2}{3}\times0\right)=1$$
$$\frac{2}{3}-\left(\frac{2}{3}\times1\right)=0$$
$$0-\left(\frac{2}{3}\times\frac{-3}{10}\right)=0+\frac{2}{10}=\frac{1}{5}$$
$$-\frac{1}{9}-\left(\frac{2}{3}\times\frac{2}{15}\right)=-\frac{1}{9}-\frac{4}{45}=\frac{-5-4}{45}=-\frac{9}{45}=-\frac{1}{5}$$
$$\frac{10}{9}-\left(\frac{2}{3}\times\frac{16}{5}\right)=\frac{10}{3}-\frac{32}{15}=\frac{50-32}{15}=\frac{18}{15}=\frac{6}{5}$$

$20\times\frac{16}{5}+15\times\frac{6}{5}=82$

Next find $C_j - Z_j$

Next Check the optimality condition $C_j - Z_j$ is either 0 or ≥ 0. In this Iteration we get either 0 or $0 \geq 0$. ~~The so~~ So we get the optimal solution. The soln is 82.

0
18
64
82

$$\bar{X}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{matrix}0 \text{ (No value)}\\6/5\\16/5\end{matrix}$$

∴ Min (Z) = 82

∴ Objective fn $Z = 7x_1 + 15x_2 + 20x_3$

$= 7(0) + 15(6/5) + 20(16/5) = 82$ Checked

Linear Programming

The Simplex Method:-

When most real-life problem are formulated as an LP model more than two variables are considered and are too large for the graphical method of soln. G.B. Dantzig in 1947 gave optimal soln, a more efficient method for such problems, called the Simplex Method.

Q1 What is a 'Simplex' in one, two, three and n-dimensions?

Ans

1. In one dimension, a simplex is a line segment connecting two points.
2. In two dimension, a simplex is a triangle formed by joining three points.
3. In three dimensions, it is a four sided pyramid having four corners.

(4). In n-dimensional space a simplex is an n-dimensional object connecting n+1 points.

Q2 What is difference between graphical and simplex method?

Ans · Concepts are similar to both the methods:

In graphical method, extreme points of the feasible solution space are examined to search for optimal solution at one of them.

Whereas in L.P problems with several variables, we may not be able to graph the feasible region but the optimal solution will still lie at an extreme point of the many sided, multidimensional figure (called as n-dimensional polyhedron) that represents the feasible solution space.

S.M. examines the extreme points in a systematic manner repeating the same set of steps of the algorithm until an optimal soln is reached. So it is also called Iterative Method.

The Method consists in moving from one iteration (extreme point) to another. for optimal soln in a finite no. of steps and also indicates unbounded soln if any.

$\Delta_3 = c_3 - c_B Y_3 = 4 - (0,0,0)(0,5,4) -$

Simplex Method :-

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Introduction.

The simplex method is an iterative (step-by-step) procedure by which a new basic feasible solution can be obtained from a given (initial) basic feasible solution so that the value of the objective function is improved.

The general LPP can be converted in std. matrix form.

Max $z = \vec{C}\vec{X}$ subject to
$\vec{A}\vec{X} = \vec{b}$, $\vec{X} \geq 0$.

Where the system $\vec{A}\vec{X} = \vec{b}$ consists of m linear equations in $N \geq m$ decision variables (containing original(n), slack(l) and surplus(m) variables); and

$\vec{C} = (c_1, c_2, \cdots, c_N)$, $\bar{X} = (x_1, x_2, \cdots x_N)$,
$\vec{b} = (b_1, b_2, \cdots, b_m)$ and

$$\bar{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \hline a_{m1} & a_{m2} & \cdots & a_{mN} \end{bmatrix}_{m \times N}$$

Note Column vectors will also be represented by row vectors without using transpose symbol (T)
Scalar Multiplication of the vectors

Clearly $N = l + m + n$
(l → slack, m → surplus, n → original variable)

Theory of S.M

Some definitions & notations

Max $z = \bar{C}\bar{X}$, subject to
$\bar{A}\bar{X} = \bar{b}$; $X \geq 0$.

First we denote the j^{th} column of $m \times N$ matrix A by $\vec{a}_j$, $j = 1, 2 \ldots N$ so that

$$A = \{a_1, a_2, \cdots a_N\} \quad \text{—— (1)}$$

Now form $m \times m$ non-singular submatrix $\vec{B}$ called the basis matrix, whose column vectors are m no. of linearly independent columns selected from matrix A given by (1) and re-named as $\beta_1, \beta_2 \ldots \beta_m$

$$\therefore \vec{B} = \{\beta_1, \beta_2 \cdots \beta_m\} \quad \text{—— (2)}$$

The columns of $\vec{B}$ form basis for R^m.

Now any column a_j of A can be represented by a linear combinations of columns of B. Following notations will be adopted for such a linear combination

$$a_j = x_{1j}\beta_1 + x_{2j}\beta_2 + \cdots + x_{mj}\beta_m$$
$$= (\beta_1, \beta_2, \cdots \beta_m)(x_{1j}, x_{2j}, \cdots x_{mj})$$

or $\vec{a}_j = \vec{B}\vec{x}_j$, where, (89)

$x_j = (x_{1j}, x_{2j} \dots x_{mj})$.

$\therefore \vec{x}_j = B^{-1}\vec{a}_j$ ——— (3)

where x_{ij} $(i=1,2,\dots,m)$ are the scalars required to express a_j in such a form.

The vector $\vec{x}_j$ will change if the colums of A forming $\vec{B}$ change.

Any basis matrix B will yield a basic solution to $\bar{A}\bar{X} = \bar{b}$. This sol^n may be denoted by m-component vector as

$$\bar{X}_B = (x_{B1}, x_{B2}, \dots, x_{Bm})$$

when $\bar{X}_B$ is determined from

$$\boxed{\bar{X}_B = B^{-1}\bar{b}} \text{ ——— (4)}$$

The subscript i in x_{Bi} shows that variable x_r corresponds to column β_i of basis matrix $\vec{B}$. This, however, does not show which variable of $(\vec{A}\vec{x} = \vec{b})$ is x_{Bi}. This also recalls that variables $x_{B1}, x_{B2} \dots x_{Bm}$ are called "basic variables" and the remaining n-variables are called non-basic variables.

Corresponding to any $\bar{X}_B$, the m component row vector (CB) containing the constraints taken from the objective f^n.

$z = c_1x_1 + c_2x_2 + \dots + c_Nx_N$ is associated with the basic variables that is

$C_B = (C_{B1}, C_{B2}, \dots, C_{Bm})$

the subscript i shows that C_{Bi} is the coefficient of the basic variable x_{Bi} in the objective function.

The notation implies that for any basic feasible sol^n, since all non-basic variables are zero, the value of the objective f^n z is given by

$z = C_{B1}X_{B1} + C_{B2}X_{B2} + \dots + C_{Bm}X_{Bm} + 0$

(Since all remaining N−m variables are non-basic and hence zero).

$= (C_{B1}, C_{B2}, \dots, C_{Bm})(X_{B1}, X_{B2}, \dots X_{Bm})$

$$\therefore \boxed{z = C_B X_B} \text{ ——— (5)}$$

In linear programming terminology the constant vector C_B is called the "price vector" and this is how one refers them as follows.

Finally a new variable z_j is defined as follows:—

Δ_{18}. $\Delta_3 = c_3 - c_B Y_3 = 4 - (0,0,0)(0,5,4) = 4$

$$z_j = x_{1j} c_{B1} + x_{2j} c_{B2} + \dots + x_{mj} c_{Bm}$$

$$= \sum_{i=1}^{m} c_{Bi}\, x_{ij} \quad \text{——— (6a)}$$

$$= (c_{B1}, c_{B2}, \dots, c_{Bm})(x_{1j}, x_{2j}, \dots, x_{mj})$$

$$\boxed{\text{i.e. } z_j = c_B x_j} \quad \text{——— (6b)}$$

There exist z_j for each a_j, then z_j corresponding to a_j changes as the columns of A forming $\bar{B}$ change. This variable z_j will assume special importance in the subsequent analysis.

Ex: Illustrate defn & notations by the following programming problem.

Maximize $z = x_1 + 2x_2 + 3x_3 + 0x_4 + 0x_5$, s.t.

$$4x_1 + 2x_2 + x_3 + x_4 = 4$$
$$x_1 + 2x_2 + 3x_3 + 0 - x_5 = 8.$$

Soln. First of all constraints eqn in matrix form may be written as. $\boxed{A\ X = b.}$

$$\begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 4 & 2 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 & -1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

or $\bar{A}\bar{X} = \bar{b}$

A basis matrix $\bar{B} = (\beta_1, \beta_2)$ is formed using columns a_3 and a_1 so that

$$\bar{\beta}_1 = \bar{a}_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \bar{\beta}_2 = \bar{a}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

The rank of matrix A is 2 and hence $\bar{a}_3, \bar{a}_1$ column vectors are linearly independent, and thus forms a basis for R^2.

Thus basis matrix is

$$B = (\beta_1, \beta_2) = \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix} \quad \begin{matrix} a_3 & a_1 \end{matrix}$$

using eqn (4), the basic feasible soln is

$$x_B = B^{-1} b = \left[\frac{1}{|B|} \operatorname{adj}(B)\right] b$$

$$= -\frac{1}{11}\begin{bmatrix} 1 & -4 \\ -3 & 1 \end{bmatrix}\begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 28 \\ 4 \end{bmatrix}$$

or $x_B = \begin{bmatrix} 28/11 \\ 4/11 \end{bmatrix} = \begin{bmatrix} x_{B1} \\ x_{B2} \end{bmatrix}$

There fore basic variables are $x_{B1} = 28/11 = x_3$

$x_{B2} = 4/11 = x_1$, and remaining variables are non-basic (which are always zero), i.e. $x_2 = x_4 = x_5 = 0$.

Also c_{B1} = Coeff. of x_{B1} = Coeff of $x_3 = c_3 = 3$

c_{B2} = Coeff of x_{B2} = Coeff of $x_1 = c_1 = 1$

Hence $c_B = (3, 1)$

Now, using (5) the value of the objective fn is

$$z = c_B x_B = (3, 1)\begin{pmatrix} 28/11 \\ 4/11 \end{pmatrix} = \frac{88}{11}$$

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Also, any vector $\bar{a}_j$ = (j=1,2,3,4,5) can be expressed as a linear (9.1) combination of vectors $\bar{\beta}_i$ (i=1,2). Therefore, to express $\bar{a}_2$ as a linear combination of β_1, β_2, we have

$$\bar{a}_2 = x_{12}\bar{\beta}_2 + x_{22}\bar{\beta}_1$$
$$= x_{12}\bar{a}_3 + x_{22}a_1$$

To compute values of scalars x_{12} and x_{22}, use the result (B) to get

$$x_2 = B^{-1}a_2 = -\frac{1}{11}\begin{pmatrix} 1 & -4 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$$

Therefore $x_{12} = 6/11$, $x_{22} = 4/11$

Similar treatment can be adopted expressing other $\bar{a}_j$'s as linear combinations of β_1 and β_2.

Now using (6.3) the variable z_2 corresponding to vector a_2 can be obtained as

$$z_2 = c_B x_2 = (3,1)\begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \left(3\frac{6}{11} + 1\frac{4}{11}\right) = \frac{22}{11}$$

Similarly z_1, z_3, z_4, z_5 can also be computed

$\Delta_3 = c_3 - c_B Y_3 = 4 - (0,0,0)(0,5,4) = 4$